

A NOTE ON “FOLDING WHEELS AND FANS”

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Abstract. In [6] Gervacio, Guerrero and Rara obtained formulas for the size of the largest clique onto which a graph G folds. We prove an interpolation lemma which simplifies some of their arguments and we correct a mistake in their formula for wheels.

1 Introduction

We consider undirected graphs without loops or multiple edges.

Definition 1. Let $G = (V, E)$ be a graph and let x and y be two vertices in G that are at distance two. A simple fold with respect to x and y is the operation which identifies x and y .

If multiple edges appear then they are identified.

When G is connected then any maximal sequence of simple folds turns G into a clique. Cook and Evans, and later also Wood, show that a connected graph G can always be folded onto a clique with $\chi(G)$ vertices [3,14]. We are interested in the number $\Sigma(G)$ of vertices of the largest clique onto which a connected graph G folds.

We follow the terminology of [7]. An epimorphism is a surjective homomorphism. Consider an epimorphism $\phi : G \rightarrow K_s$. We call

$$\{ \phi^{-1}(h) \mid h \in V(K_s) \}$$

the color classes of ϕ in G . A homomorphism $\phi : G \rightarrow H$ is faithful if $\phi(G)$ is an induced subgraph of H . A faithful epimorphism is called complete.

Definition 2. Let G be a graph. The achromatic number $\Psi(G)$ is the number of vertices in a largest clique K for which there is a complete homomorphism

$$f : G \rightarrow K.$$

Equivalently, $\Psi(G)$ is the maximal number of colors in a proper vertex coloring of G such that for every pair of colors there is an edge of which the endvertices are colored with the two colors. Finding the achromatic number of a graph is NP-complete, even for trees, however it is fixed-parameter tractable [4,11,13].

Lemma 1. *Assume that G has a universal vertex u . Then*

$$\Sigma(G) = 1 + \Psi(G - u) = \Psi(G).$$

Proof. Any two nonadjacent vertices of $G - u$ are at distance two in G . Thus any achromatic coloring of G is obtained by a maximal series of folds. The universal vertex must appear in a color class by itself.

Harary and Hedetniemi showed that, if G is the join of two graphs G_1 and G_2 , then $\Psi(G) = \Psi(G_1) + \Psi(G_2)$ [8]. \square

Bodlaender showed that achromatic number is NP-complete for trivially perfect graphs [1] (see also [4]). Since the class of trivially perfect graphs is closed under adding a universal vertex, by Lemma 1 also the folding problem is NP-complete for this class of graphs.

2 An interpolation theorem

Gervacio et al [6, Theorem 4.2 and Theorem 5.2] showed that, for fans and wheels, there is a fold onto K_k for any

$$\chi(G) \leq k \leq \Sigma(G).$$

The following theorem is similar to the interpolation theorem of Harary, Hedetniemi and Prins [9].

Theorem 1. *Let G be a connected graph and let $\chi(G) \leq k \leq \Sigma(G)$. There is a folding of G onto K_k .*

Proof. The proof is similar to the one given in [7, Proposition 4.12] for the achromatic number.

Consider a simple fold, which maps G to G' . Then (see eg [7, Corollary 4.6]),

$$\chi(G) \leq \chi(G') \leq \chi(G) + 1.$$

The folding of G onto the complete graph with $\Sigma(G)$ vertices is a sequence of simple folds. Therefore, after some initial sequence of simple folds, there must appear a graph G'' with chromatic number k . There is a fold of G'' onto K_k [3,14]. The concatenation gives the desired folding of G onto K_k . \square

3 Wheels

Consider C_9 and label the vertices $[1, \dots, 9]$. We obtain a complete coloring with four colors a, b, c and d by coloring the vertices in order

$$[a, d, b, a, c, d, a, c, b].$$

Thus $\Psi(C_9) \geq 4$. However, the formula in [6, Theorem 6.1] gives $\Psi(C_9) = 3$ (for $s = 1$ and $n = 2(s + 1)^2 + 1 = 9$).

The following result of Marcu provides the upperbound. It shows that $n \geq 10$ when $\Psi(C_n) = 5$. See also [5,10].

Theorem 2 ([12]). *For an n -cycle with achromatic number Ψ ,*

$$n \geq \begin{cases} \frac{\Psi(\Psi-1)}{2} & \text{if } \Psi \text{ is odd} \\ \frac{\Psi^2}{2} & \text{if } \Psi \text{ is even.} \end{cases}$$

3.1 Threshold graphs

A graph is trivially perfect if it has no induced C_4 and no induced P_4 . The result of Bodlaender implies that the folding number is NP-complete for trivially perfect graphs [1]. In the following theorem we show that if in every induced subgraph there is at most one component with more than one vertex, then the problem is polynomial.

For our purposes the following characterization of threshold graphs is most useful.

Theorem 3 ([2]). *A graph is a threshold graph if and only if every induced subgraph has an isolated vertex or a universal vertex.*

Alternatively, threshold graphs are those graphs without induced P_4 , C_4 and $2K_2$.

Theorem 3 immediately gives the following result.

Theorem 4. *If G is a threshold graph then*

$$\chi(G) = \Sigma(G) = \Psi(G).$$

Proof. By a result of Harary and Hedetniemi [8, Proposition 6], when G is the join of two graphs G_1 and G_2 then

$$\Psi(G) = \Psi(G_1) + \Psi(G_2). \tag{1}$$

Now assume that G has an isolated vertex x . We claim that

$$\Psi(G) = \max \{ 1, \Psi(G - x) \}. \tag{2}$$

To see that, observe that x cannot form a color class by itself. \square

References

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